

# The degree of point configurations: from Ehrhart theory to almost neighborly polytopes

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## Abstract

The degree of a point configuration is defined as the maximal codimension of its interior faces. This concept is motivated from a corresponding Ehrhart-theoretic notion for lattice polytopes and is related to neighborly polytopes, the generalized lower bound theorem and Tverberg theory.

The main results of this paper are a complete classification of point configurations of degree 1, as well as a structure result on point configurations whose degree is less than a third of the dimension. Statements and proofs involve the novel notion of a weak Cayley configuration.

## 1 Introduction and motivation

### 1.1 Introduction

In this paper we introduce and discuss an invariant of a point configuration which we hope to be of interest to researchers in geometric combinatorics. The idea is to transfer the intuition acquired over the last years in the investigation of an Ehrhart-theoretic invariant of a lattice polytope (called its degree) to a more general and combinatorial setting.

Let us explain this briefly. Given a lattice polytope  $P$  (a polytope with vertices in the lattice  $\mathbb{Z}^d$ ), the Ehrhart polynomial counts the number of lattice points in dilates of  $P$ . As it turns out, the complexity of the Ehrhart polynomial is directly related to the largest natural number  $k$  such that  $kP$  has no interior lattice points. Now, here is our naive observation: this clearly implies that any set of  $k$  vertices of  $P$  has to lie in a common facet, since otherwise their sum would lie in the interior of  $kP$ . So, the following natural question arises: How strong is such a combinatorial obstruction?

In geometric combinatorics such questions were studied quite intensively, in particular for simplicial polytopes. Here, the crucial notion is that of a  $k$ -neighborly polytope,

where any set of  $k$  vertices has to be the vertex set of a face. Thus, we are led to consider the following generalization (which already appeared in the exercises of Grünbaum's book [14]).  $P$  is *k-almost neighborly*, if any set of  $k$  vertices is the subset of the vertex set of a facet. Note that for simplicial polytopes these two notions agree.

In this paper, we hope to shed some light on the structure of almost neighborly polytopes, even if not all questions can yet be fully answered. For instance, if a polytope is  $> \lfloor \frac{d}{2} \rfloor$ -neighborly, then it has to be a simplex. What happens for almost neighborly polytopes?

The first part of this paper contains the summary of our main results and their interpretations in different contexts. As it turns out, it is convenient to consider above questions for arbitrary point configurations  $A$  instead of just vertex sets of polytopes. Moreover, as will be explained below, Ehrhart theory motivates to focus on the number  $d - k$  (which will be called the degree of  $A$ ) rather than on  $k$ . In the second part we provide the proofs for our results. These are based on a natural Gale dual interpretation of the degree.

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## 1.2 The main notions and results

Let  $A$  be a finite point configuration in  $\mathbb{R}^d$ . We will always require that  $A$  is full-dimensional (i.e., its affine span equals  $\mathbb{R}^d$ ), and we will allow that  $A$  contains repeated points. We say that a non-empty subset  $S \subset A$  is an *interior face* of  $A$  if  $\text{conv}(S)$  does not lie on the boundary of  $\text{conv}(A)$ . That is, if  $\text{conv}(S) \cap \text{int}(\text{conv}(A))$  is not empty. Recall that a *facet* of a polytope is a codimension one face. Here are our main definitions:

**Definition 1.1.** The *degree*,  $\deg(A)$ , is the maximal codimension of an interior face of  $A$ . The *codegree* of  $A$  is given as  $\text{codeg}(A) := d + 1 - \deg(A)$  and equals the maximal positive integer  $c$  such that every subset of  $A$  of size  $< c$  is a subset of the set of vertices of a facet of  $\text{conv}(A)$ . We define the *degree (resp. codegree) of a convex polytope  $P$*  as the degree (resp. codegree) of its set of vertices,  $\deg(\text{vert}(P))$  (resp.  $\text{codeg}(\text{vert}(P))$ ). Note that the degree of a polytope only depends on its combinatorial type.

In particular,  $0 \leq \deg(A) \leq d$ . In this paper we hope to convince the reader that the degree of a point configuration is a natural and worthwhile invariant to study. Before giving evidence for this claim by relating the degree to some existing notions, we would like to state our main results. For this we need the following novel definition:

**Definition 1.2.** A point configuration  $A$  is a *weak Cayley configuration* of length  $k$  if there exists a partition  $A = A_0 \uplus A_1 \uplus \dots \uplus A_k$  such that for any  $\{0\} \subset I \subsetneq \{0, 1, \dots, k\}$ ,  $\text{conv}(\bigcup_{i \in I} A_i)$  is a proper face of  $\text{conv}(A)$ . The sets  $A_1 \dots A_k$  are called the *factors* of the configuration. Note that while  $A_0$  may be the empty set, the factors  $A_1, \dots, A_k$  have to be non-empty.

Weak Cayley configurations of large length are examples of point configurations with small degree.

**Proposition 1.3.** Let  $A \subset \mathbb{R}^d$  be a weak Cayley configuration of length  $k$ , then  $\deg(A) \leq d + 1 - k$ .

The main results of this paper may be summarized as follows:

**Theorem 1.4.** Let  $A$  be a point configuration in  $\mathbb{R}^d$  of degree  $\delta$ .

- (I) If  $\delta = 0$ , then  $A$  is the set of vertices of a  $d$ -simplex (possibly with repetitions).
- (II) If  $\delta = 1$ , then
  - (a)  $d = 1$  and  $\text{conv}(A)$  is an interval that contains at least an interior point of  $A$ , or
  - (b)  $d \geq 2$  and  $\text{conv}(A)$  is a  $k$ -fold pyramid over a two-dimensional point configuration without interior points, or
  - (c)  $d \geq 3$  and  $\text{conv}(A)$  is a  $k$ -fold pyramid over a prism over a simplex with the non-vertex points of  $A$  all on the “vertical” edges, or
  - (d)  $d \geq 3$  and  $\text{conv}(A)$  is a simplex with a vertex  $v$  and the other points of  $A$  on the edges adjacent to  $v$ .

See Figure 1 for some examples.

- (III) Let  $r := n - d - 1$ . If

$$d \geq r + 2\delta,$$

then  $A$  is a pyramid.

- (IV) If  $d > 3\delta$ , then  $A$  is a weak Cayley configuration of length at least  $d - 3\delta + 1$ .

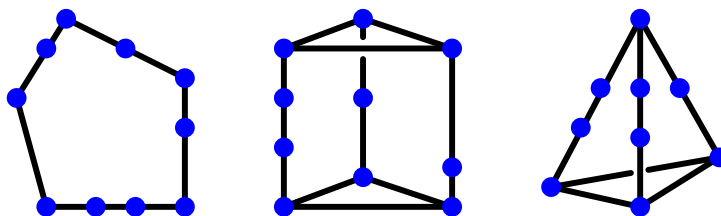


Figure 1: Some point configurations with degree 1

All proofs will be given in the second part. Specifically, Section 2.2 contains the proofs of (II) and (III), while Sections 2.4 and 2.5 provide proofs of (II) and (IV), respectively.

Result (IV) should be seen as a converse statement to Proposition 1.3. The reader may also have noticed that the assumption in (IV) might be strengthened for  $\delta = 1$ . Indeed, if  $d > 2$  and  $\delta = 1$ , then  $A$  is a weak Cayley configuration of length at least  $d - 1$ . This observation (among others, as will be explained below) motivates our main conjecture:

**Conjecture 1.5.** *Let  $A$  be a  $d$ -dimensional point configuration of degree  $\delta$ . If  $d > 2\delta$ , then  $A$  is a weak Cayley configuration of length  $d + 1 - 2\delta$ .*

The conjectured bound, if correct, is sharp. For example, an even-dimensional cyclic  $d$ -polytope with  $2d + 1$  vertices has degree  $\delta = \frac{d}{2}$  but is not a weak Cayley configuration of length  $\geq 2$ . See Example 2.24 for details.

### 1.3 Relation to concepts in geometric combinatorics

#### 1.3.1 Almost neighborly polytopes

Recall that a  $d$ -polytope  $P$  is  $k$ -neighborly, if every subset of vertices of  $P$  of size  $\leq k$  is the set of vertices of a face of  $P$ . The following well-known result (see, for example [14, Chapter 7]) motivates the definition of a  $d$ -polytope as *neighborly*, if it is  $\lfloor \frac{d}{2} \rfloor$ -neighborly.

**Theorem 1.6.** *If a  $d$ -polytope  $P$  is  $k$ -neighborly for any  $k > \lfloor \frac{d}{2} \rfloor$ , then  $P$  must be the  $d$ -dimensional simplex.*

Neighborly polytopes form a well-studied family of polytopes [28]. One of the main reasons of this interest is McMullen's Upper Bound Theorem [21]: The number of  $i$ -dimensional faces of a  $d$ -polytope  $P$  with  $n$  vertices is maximal for neighborly polytopes.

In the definition of  $k$ -neighborly, one can relax the condition of being the set of vertices of a face by belonging to the set of vertices of a facet: we say that a  $d$ -dimensional point configuration  $A$  is  $k$ -almost neighborly if every subset of  $A$  of size  $\leq k$  lies in a common facet of  $\text{conv}(A)$ . The name ‘almost neighborly’ was coined by Grünbaum in [14, Exercices 7.3.5 and 7.3.6]. According to him, this notion was already considered by Motzkin under the name of  $k$ -convex sets [24]. In [9] Breen proved that a point configuration is  $k$ -almost neighborly if and only if all its subconfigurations of size  $\leq 2d + 1$  are.

In our notation, a configuration  $A$  is  $k$ -almost neighborly if and only if  $\text{codeg}(A) > k$ . In particular, Theorem 1.4(II) classifies  $(d - 1)$ -almost neighborly point configurations, and Theorem 1.4(III) states that any  $k$ -almost neighborly point configuration with less than  $2(k + 1)$  points must be a pyramid. Moreover, Theorem 1.4(IV) gives an explicit structure result for  $k$ -almost neighborly point configurations with  $k > \frac{2}{3}d$ . Our main conjecture, Conjecture 1.5, would extend this to  $k > \frac{d}{2}$ . Hence, this can be seen as a potentially precise analogue of Theorem 1.6 for almost neighborly point configurations.

The concept of almost neighborliness is strongly related to the concept of weakly neighborliness [5]. In particular, in [5, Theorem 15] Bayer already classified 3-dimensional polytopes  $P$  with  $\deg(\text{vert}(P)) = 1$  as prisms over simplices and pyramids over polygons.

### 1.3.2 Triangulations and the Generalized Lower Bound Theorem

Let  $\mathcal{T}$  be a  $(d-1)$ -dimensional simplicial complex. If  $f_i(\mathcal{T})$  denotes its number of  $i$ -dimensional faces, then the  $h$ -vector  $(h_0(\mathcal{T}), \dots, h_d(\mathcal{T}))$  is defined by the polynomial relation

$$\sum_{i=0}^d h_i(\mathcal{T}) t^i = \sum_{i=0}^d f_{i-1}(\mathcal{T}) t^i (1-t)^{d-i}.$$

This polynomial is called the  $h$ -polynomial  $h_{\mathcal{T}}(t)$  of  $\mathcal{T}$ .

By the famous  $g$ -theorem [7, 33],  $h$ -polynomials of the boundary complex of simplicial  $d$ -polytopes  $P$  are completely known. In particular,  $h_{\partial P}(t)$  has degree  $d$ , it satisfies the Dehn-Sommerville equations  $h_i(\partial P) = h_{d-i}(\partial P)$ , and it is unimodal (i.e.,  $h_i(\partial P) \geq h_{i-1}(\partial P)$  for all  $1 \leq i \leq \lfloor d/2 \rfloor$ ). In 1971, McMullen and Walkup [23] posed the following famous conjecture regarding its unimodality, which is known as the Generalized Lower Bound Conjecture (GLBC):

**Theorem 1.7** (Generalized Lower Bound). *Let  $P$  be a simplicial  $d$ -polytope. For  $i \in \{1, \dots, \lfloor d/2 \rfloor\}$ ,  $h_i(\partial P) = h_{i-1}(\partial P)$  if and only if  $P$  can be triangulated without interior faces of dimension  $\leq d-i$ .*

Here, we always assume that triangulations  $\mathcal{T}$  of  $d$ -polytopes  $P$  have only vertices of  $P$  as vertices. The conjecture had remained open until it was proved very recently by Murai and Nevo [25].

It is instructive to reformulate the previous theorem. For this let us consider a triangulation  $\mathcal{T}$  of an arbitrary  $d$ -polytope  $P$ . An *interior face* of  $\mathcal{T}$  is a face of  $\mathcal{T}$  that is not contained in a facet of  $P$ . In this situation, the degree of the  $h$ -polynomial of  $\mathcal{T}$  is well-known, see [22, Prop. 2.4] or [10, Corollary 2.6.12].

**Proposition 1.8.** *Let  $\mathcal{T}$  be a triangulation of a polytope. Then  $\deg(h_{\mathcal{T}}(t))$  equals the maximal codimension of an interior face of  $\mathcal{T}$ .*

Considering again a simplicial  $d$ -polytope  $P$  one defines  $g_0(\partial P) := 1$ , and  $g_i(\partial P) = h_i(\partial P) - h_{i-1}(\partial P)$  for  $i = 1, \dots, \lfloor \frac{d}{2} \rfloor$ . They form the coefficients of the so-called  $g$ -polynomial  $g_{\partial P}(t)$ . Therefore, Theorem 1.7 yields for a simplicial polytope  $P$ :

$$\deg(g_{\partial P}(t)) = \min(\deg(h_{\mathcal{T}}(t)) : \mathcal{T} \text{ triangulation of } P).$$

In other words, the degree  $s$  of the  $g$ -polynomial of a simplicial polytope certifies the existence of *some* triangulation that avoids interior faces of dimension  $\leq d-1-s$ . Equivalently, the simplicial polytope  $P$  is called  $s$ -stacked [23].

For general polytopes, it is also possible to define (toric)  $h$ - and  $g$ -polynomials [35]. In this case, by [36] any rational polytope  $P$  (conjecturally any polytope) satisfies

$$\deg(g_{\partial P}(t)) \leq \min(\deg(h_{\mathcal{T}}(t)) : \mathcal{T} \text{ triangulation of } P).$$

It is known that a simplex is the only polytope satisfying  $\deg(g_{\partial P}(t)) = 0$ . Note that the previous inequality may not be an equality. For instance, a 3-polytope  $P$  which is a prism over a pentagon satisfies  $\deg(h_{\mathcal{T}}(t)) = 2$  for any triangulation while  $\deg(g_{\partial P}(t)) = 1$ . In this general situation, it is a hard, open problem to classify all polytopes with  $\deg(g_{\partial P}(t)) = 1$  (these polytopes are called *elementary*, see Section 4.3 in [19]).

To describe how the results of this paper fit into this framework, let us consider the degree of the vertex set  $\text{vert}(P)$  of a  $d$ -polytope  $P$ . By observing that any interior simplex  $S$  of  $\text{vert}(P)$  can be extended to a triangulation that uses  $S$  as a face, we see that  $\deg(\text{vert}(P))$  is the maximal codimension of an interior simplex of some triangulation of  $P$ . In other words,

$$\deg(\text{vert}(P)) = \max(\deg(h_{\mathcal{T}}(t)) : \mathcal{T} \text{ triangulation of } P).$$

Hence, classifying polytopes of degree  $\delta$  is equivalent to studying polytopes where *all* triangulations avoid interior faces of dimension  $\leq d - 1 - \delta$ . This problem is more tractable than the one described above, and Theorem 1.4(II) solves it for  $\delta = 1$ .

Finally, a particular motivation for the study of point configurations of degree 1 comes from the *Lower Bound Theorem* for balls (see [10, Theorem 2.6.1]). It states that if  $A$  is a  $d$ -dimensional configuration of  $n$  points, then any triangulation using all the points in  $A$  uses at least  $(n - d)$  full-dimensional simplices. Hence,  $\deg(A) = 1$  holds precisely when *all* triangulations using all the points of  $A$  have size  $(n - d)$ . This reflects the fact that all triangulations of  $A$  are stacked. This interpretation of Theorem 1.4(II) is used in [8] to derive results in additive combinatorics.

### 1.3.3 Totally splittable polytopes

A *split* of a polytope is a subdivision with exactly two maximal cells, which are separated by a *split hyperplane*. A polytope  $P$  is called *totally splittable*, if each triangulation of  $P$  is a common refinement of splits. In [16, Theorem 9], Herrmann and Joswig establish a complete classification of totally splittable polytopes: simplices, polygons, prisms over simplices, crosspolytopes and a (possible multiple) join of these.

Two splits of  $P$  are called *compatible* if their split hyperplanes do not intersect in the interior of  $P$ . We say that  $P$  is *strongly totally splittable* if any triangulation is a common refinement of compatible splits. The discussion of the previous section leads to the following result whose proof is left to the reader.

**Proposition 1.9.**  *$P$  is strongly totally splittable if and only if  $\deg(\text{vert}(P)) \leq 1$ .*

As a corollary, every polytope of degree 1 is totally splittable. In particular, by analyzing the cases of Herrmann and Joswig's result one can deduce a proof of Proposition 2.33 for the case that the points in  $A$  are in convex position.

### 1.3.4 Tverberg's Theory

Let  $A$  be a configuration of  $n$  points in  $\mathbb{R}^d$ . We say that  $x \in \mathbb{R}^d$  is in the  *$m$ -core* of  $A$ , denoted by  $\mathcal{C}_m(A)$ , if every closed halfspace containing  $x$  also contains at least  $m$  points

of  $S$ . We say that a point  $x \in \mathbb{R}^d$  is an  $m$ -divisible point of  $A$ , if there exist  $m$  disjoint subsets  $S_1, \dots, S_m$  of  $A$  such that  $x \in \text{conv}(S_i)$  for  $i = 1, \dots, m$ . We denote by  $\mathcal{D}_m(A)$  the set of  $m$ -divisible points of  $A$ . The well-known Tverberg's Theorem asserts that  $\mathcal{D}_m(A) \neq \emptyset$  whenever  $n \geq (m-1)(d-1) + 1$ . A good introduction for these concepts can be found in [20, Chapter 8].

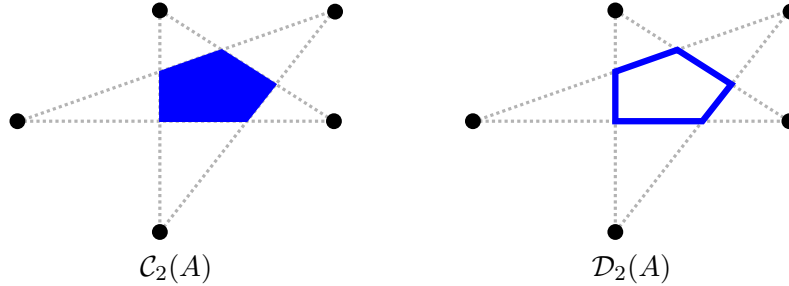


Figure 2: When  $A$  is the vertex set of a pentagon,  $\mathcal{C}_2(A)$  is the inner pentagon delimited by the interior diagonals, while  $\mathcal{D}_2(A)$  is only the boundary of this inner pentagon.

It is easy to see that  $\text{conv}(\mathcal{D}_m(A)) \subset \mathcal{C}_m(A)$ . Equality was conjectured [29, 31], and actually holds when  $d = 2$  or  $m = 1$ . However, Avis found a counterexample for  $n = 9$ ,  $d = 3$  and  $m = 3$  [1], and Onn provided a systematic construction for counterexamples [27].

The proof of Theorem 1.4(IV) yields the following result (see Observations 2.8 and 2.25):

**Corollary 1.10.**  $\mathcal{C}_m(A) \subset \mathcal{D}_{3m-2(n-d)}(A)$ .

Note that this result is only non-trivial, if  $m > \frac{2}{3}(n-d)$ . On the other hand,  $\mathcal{C}_m(A) \neq \emptyset$  implies  $m \leq n-d$ . Hence, Corollary 1.10 is of interest for configurations that admit points in some  $m$ -core with large  $m$ . In this context our main conjecture, Conjecture 1.5, is equivalent to  $\mathcal{C}_m(A) \subset \mathcal{D}_{2m-(n-d)}(A)$ .

## 1.4 The relation to Ehrhart theory

### 1.4.1 The lattice degree of a lattice polytope

Let us consider the situation where  $P \subset \mathbb{R}^d$  is a *lattice polytope*, i.e., its vertices are in the lattice  $\mathbb{Z}^d$ . We will identify lattice polytopes up to unimodular equivalence, i.e., affine isomorphisms of the lattice. Due to Ehrhart [13] the generating function enumerating the number of lattice points in multiples of  $P$  is a rational function of the following form:

$$\sum_{k \geq 0} |(kP) \cap \mathbb{Z}^d| t^k = \frac{h_P^*(t)}{(1-t)^{d+1}},$$

where the polynomial  $h_P^*(t) = \sum_{i=0}^d h_i^* t^i$  is called the  $h^*$ -polynomial (or  $\delta$ -polynomial) of  $P$  (see [2, 3, 17, 37]). Stanley [32, 34] showed that the coefficients of  $h_P^*$  are non-negative integers. Ehrhart theory can be understood as the study of these coefficients.

The degree of  $h_P^*(t)$ , i.e., the maximal  $i \in \{0, \dots, d\}$  with  $h_i^* \neq 0$ , is called the (lattice) degree  $\deg_{\mathbb{Z}}(P)$  of  $P$  [3]. The (lattice) codegree of  $P$  is given as  $\text{codeg}_{\mathbb{Z}}(P) := d + 1 - \deg_{\mathbb{Z}}(P)$  and equals the minimal positive integer  $k$  such that  $kP$  contains interior lattice points. In recent years these notions and their (algebro-)geometric interpretations have been intensively studied [2, 3, 4, 11, 12, 15, 26].

The notion of the degree of a lattice polytope was defined in [3], where it was noted that  $\deg_{\mathbb{Z}}(P)$  should be considered the “lattice dimension” of  $P$ . This interpretation of the degree was motivated by the following three basic properties:

- (i)  $\deg_{\mathbb{Z}}(P) = 0$  if and only if  $P$  is unimodularly equivalent to the *unimodular simplex*  $\text{conv}(0, e_1, \dots, e_d)$ .
- (ii) For a lattice polytope  $Q \subset P$ , we have  $\deg_{\mathbb{Z}}(Q) \leq \deg_{\mathbb{Z}}(P)$  by Stanley’s monotonicity theorem [37].
- (iii) If  $P$  is a *lattice pyramid* over  $Q$  (i.e.,  $P \cong \text{conv}(0, Q \times \{1\}) \subset \mathbb{R}^{d+1}$ ), then  $\deg_{\mathbb{Z}}(P) = \deg_{\mathbb{Z}}(Q)$ .

#### 1.4.2 Comparing the degree with the lattice degree

It was already noted in [3, Prop. 1.6] that a lattice  $d$ -polytope  $P$  satisfies

$$\deg(P \cap \mathbb{Z}^d) \leq \deg_{\mathbb{Z}}(P). \quad (1)$$

If  $P$  is normal (i.e., any lattice point in  $kP$  is the sum of  $k$  lattice points in  $P$ ), then  $\deg(P \cap \mathbb{Z}^d) = \deg_{\mathbb{Z}}(P)$ . However, (1) is not an equality in general, as the following example in 3-space shows:  $P = \text{conv}(0, e_1, e_2, e_1 + e_2 + 2e_3)$ . This is a so-called Reeve simplex [30]. It satisfies  $\deg(P \cap \mathbb{Z}^d) = 0$ , but  $\deg_{\mathbb{Z}}(P) = 2$ .

In the setting of Section 1.3.2, Equation (1) can also be deduced directly from a stronger result by Betke and McMullen [6]:  $h_{\mathcal{T}}(t) \leq h_P^*(t)$ , coefficientwise, for any triangulation  $\mathcal{T}$  of  $P$  whose vertices are lattice points.

From our viewpoint, the degree may be seen as a natural generalization of the Ehrhart-theoretic lattice degree. In particular, all properties of the lattice degree mentioned above also hold in the setting of point configurations  $A \subset \mathbb{R}^d$ :

- (i)  $\deg(A) = 0$  if and only if  $A$  is the vertex set of a  $d$ -simplex (Proposition 2.14).
- (ii) For  $A' \subset A$ , we have  $\deg(A') \leq \deg(A)$  (Corollary 2.10).
- (iii) If  $A$  is a *pyramid* over  $A'$  (i.e.,  $A = A' \cup \{v\}$ , and the polytope  $P = \text{conv}(A)$  is a pyramid over  $\text{conv}(A')$  with apex  $v$ ), then  $\deg(A) = \deg(A')$  (Lemma 2.16).



### 1.4.3 Cayley configurations

All parts of our main result, Theorem 1.4, are motivated by analogous statements in Ehrhart theory. In particular, the notion of a weak Cayley configuration originates in the widely used construction of Cayley polytopes. Cayley polytopes play a very important role in the study of the degree of lattice polytopes [3, 15] and the study of mixed subdivisions of Minkowski sums via the Cayley trick [18]. Let us carefully state some natural generalizations:

**Definitions 1.11.** Let  $A$  be a point configuration in  $\mathbb{R}^d$ . We say  $A$  is a

- *lattice Cayley configuration* of length  $k$ , if  $A \subset \mathbb{Z}^d$  and there is a lattice projection  $\mathbb{Z}^d \rightarrow \mathbb{Z}^{k-1}$  such that  $A$  maps onto  $\text{conv}(0, e_1, \dots, e_{k-1})$ .
- *affine Cayley configuration* of length  $k$ , if there is an affine projection  $\mathbb{R}^d \rightarrow \mathbb{R}^{k-1}$  such that  $A$  maps onto the vertex set of an  $(k-1)$ -simplex.
- *combinatorial Cayley configuration* of length  $k$ , if there exists a partition  $A = A_1 \uplus \dots \uplus A_k$ , such that for any  $\emptyset \neq I \subsetneq \{1, \dots, k\}$ ,  $\text{conv}(\bigcup_{i \in I} A_i)$  is a proper face of  $\text{conv}(A)$ .

The sets  $A_1 \dots A_k$  are called the *factors* of the configuration. Note that they have to be non-empty (because  $A_i = \emptyset$  would imply that  $\text{conv}(\bigcup_{i \in \{1, \dots, k\} \setminus \{i\}} A_i) = \text{conv}(A)$ ).

We say, a polytope  $P \subset \mathbb{R}^d$  is a (respective) *Cayley polytope*, if its vertex set is a (respective) Cayley configuration.

Obviously, ‘lattice’ implies ‘affine’ implies ‘combinatorial’. Of course, there are affine Cayley configurations that are not lattice, and there are combinatorial Cayley configurations that are not affine (e.g., the vertices of a deformed prism in  $\mathbb{R}^3$ ).

Let us point out that the term ‘combinatorial Cayley configuration’ is not ambiguous. It indeed means combinatorially equivalent to an affine Cayley configuration as the following result shows:

**Proposition 1.12.** *Every combinatorial Cayley configuration of length  $k$  is combinatorially equivalent to an affine Cayley configuration of length  $k$ .*

The proof will be given in Section 2.3.

### 1.4.4 The main motivation for Theorem 1.4

Theorem 1.4 should be seen as a combinatorial generalization of known results in the geometry of numbers of lattice polytopes. Here are the original formulations of the statements of Theorem 1.4 in the context of lattice polytopes. In the following let  $P$  be a  $d$ -dimensional lattice polytope with  $r + d + 1$  vertices and lattice degree  $\deg_{\mathbb{Z}}(P) = s$ .

- (i) As noted above,  $P$  has degree  $s = 0$  if and only if  $P$  is a unimodular simplex.

(ii) Lattice  $d$ -polytopes  $P$  of lattice degree  $s = 1$  were classified in [3]: either  $P$  is a  $(d - 2)$ -fold lattice pyramid over the lattice triangle  $\text{conv}((0, 0), (2, 0), (0, 2))$  or  $P$  is a lattice Cayley polytope of length  $d$ .

(iii) The following result was shown in [26]: If

$$d > r(2s + 1) + 4s - 2,$$

then  $P$  is a lattice pyramid over an  $(d - 1)$ -dimensional lattice polytope.

(iv) And in [15]: If  $d > f(s) := (s^2 + 19s - 4)/2$ , then  $P$  is a lattice Cayley polytope of length  $d + 1 - f(s)$ .

The reader is invited to compare these results with the combinatorial statements for arbitrary polytopes or point configurations in Theorem 1.4. As is to be expected, the assumptions in Theorem 1.4 are more general, while the conclusions are weaker. Nevertheless, the bounds in Theorem 1.4 (III, IV) are better.

*Remark 1.13.* The conclusion of statement (iii) holds for lattice simplices with  $d > 4s - 2$  (since  $r = 0$ ). It is interesting to observe that one can replace here ‘lattice simplices’ by ‘simplicial polytopes’. This follows, since a simplicial lattice polytope satisfying  $d > 2s$  has to be a lattice simplex (use Equation (1) and Theorem 1.6).

It was noted in [3] that lattice Cayley polytopes of length  $k$  have lattice degree at most  $d + 1 - k$ . In particular, lattice Cayley polytopes of large length have small lattice degree. It was asked in [3] whether there might be a converse to this. Above statement (iv) answered this question affirmatively. The assumption in (iv) is surely not sharp, it is conjectured that  $f(s) = 2s$  should suffice, see [11, 12]. Therefore, it seems at first very tempting to also conjecture the analogue statement for combinatorial types of polytopes: Namely, that for a  $d$ -dimensional (lattice) polytope  $P$ ,  $d > 2\deg(\text{vert}(P))$  would imply that  $P$  is a combinatorial Cayley polytope of length  $d + 1 - 2\deg(\text{vert}(P))$ . Note that this statement indeed holds for  $\deg(\text{vert}(P)) = 1$  by Theorem 1.4(II). However, surprisingly, this guess is wrong as the following example shows.

*Example 1.14.* Consider the  $(d + 1)$ -dimensional point configuration

$$A = \{0, 2e_1, \dots, 2e_d, e_1 + e_{d+1}, e_1 - e_{d+1}, \dots, e_d + e_{d+1}, e_d - e_{d+1}\}.$$

It is in convex position (i.e.,  $A$  is the vertex set of  $\text{conv}(A)$ ) and has degree 2. However,  $A$  is not a combinatorial Cayley configuration of length  $> 1$ .

#### 1.4.5 Weak Cayley configurations

Even if the point configuration of Example 1.14 is not a combinatorial Cayley configuration, the subsets  $B_i = \{0, 2e_i, e_i + e_{d+1}, e_i - e_{d+1}\}$  fulfill all the necessary conditions except for the disjointness. This motivates our original definition of weak Cayley configuration.

**Definition 1.15.** A point configuration  $A$  is a *weak Cayley configuration* of length  $k$  if  $A$  can be covered by subsets  $A = B_1 \cup \dots \cup B_k$ , such that for any  $\emptyset \neq I \subsetneq \{1, \dots, k\}$ ,  $\text{conv}(\bigcup_{i \in I} B_i)$  is a proper face of  $\text{conv}(A)$ .

Setting  $A_i = B_i \setminus \bigcup_{j \neq i} B_j$  for  $1 \leq i \leq k$  and  $A_0 = A \setminus \bigcup_{i=1}^d A_i$  it is not hard to prove that this definition is indeed equivalent to the stronger Definition 1.2 stated above in the introduction. We prefer Definition 1.2 since it is more restrictive and allows for the following formulation:  $A$  is a weak Cayley configuration of length  $k$  if and only if there is a (possibly empty) face  $F$  of  $\text{conv}(A)$ ,  $A_0 := F \cap A$ , such that the contraction (i.e., iterated vertex figure)  $A/A_0$  is a Cayley configuration of length  $k$ . Then the factors of  $A$  are defined as the factors of  $A/A_0$ .

Note that every combinatorial Cayley configuration is a weak Cayley configuration. Example 1.14 motivates why even for polytopes (instead of more general point configurations) it is necessary to consider weak Cayley configurations. The point configuration in this example is a weak Cayley configuration of length  $d$ , with  $A_0 = \{0\}$  and  $A_i = \{2e_i, e_i + e_{d+1}, e_i - e_{d+1}\}$ .

Summing up, Theorem 1.4(IV) should be seen as the correct combinatorial analogue of the statement (iv) for lattice polytopes in the previous section. Moreover, the conjecture that  $f(s) = 2s$  suffices in the lattice setting matches precisely Conjecture 1.5.

## 2 Proof of results

### 2.1 Gale duality

We start by recalling some basic results on Gale duality. For an introduction one can consult [20, Section 5.6] or [38, Chapter 6], where the reader can find proofs for the results presented in this section.

Gale duality transforms a configuration  $A := (a_1, \dots, a_n)$  of  $n$  points whose affine span is  $\mathbb{R}^d$  into a configuration  $V := (v_1, \dots, v_n)$  of  $n$  vectors in  $\mathbb{R}^{n-d-1}$ . The combinatorial properties of these two configurations are strongly related.

To avoid confusion, our vector configurations will always be on the Gale dual side and we will use  $r := n - d - 1$  to denote their dimension. Remark that there may be repeated vectors in  $V$ , even if all the  $a_i$  were different.

#### Notation

Let  $A = \{a_1, \dots, a_n\}$  be a  $d$ -dimensional point configuration. Its set of *affine functions* is

$$\text{a-Val}(A) = \left\{ \nu \in \mathbb{R}^n \mid \nu_i = ca_i - z, \text{ where } c \in (\mathbb{R}^d)^*, z \in \mathbb{R} \right\}.$$

In particular, the affine functions of  $A$  determine the face lattice of  $\text{conv}(A)$ .

Similarly, we define the set of *linear dependences*  $\text{Dep}(V)$  of a vector configuration  $V = \{v_1, \dots, v_n\}$  as

$$\text{Dep}(V) = \left\{ \lambda \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i v_i = \mathbf{0} \right\}.$$

The sets  $C(\lambda) := \{i \mid v_i \in V \text{ and } \lambda_i \neq 0\}$  for  $\lambda \in \text{Dep}(V)$  are called the *vectors* of  $\mathcal{M}(V)$ , the oriented matroid of  $V$ . The set of vectors of  $\mathcal{M}(V)$  is denoted by  $\mathcal{V}(V)$ . We see the vectors of  $\mathcal{M}(V)$  as signed sets, since each vector  $C := C(\lambda)$  can be decomposed into  $C^+ = \{i \mid \lambda_i > 0\}$  and  $C^- = \{i \mid \lambda_i < 0\}$ . If  $C^- = \emptyset$ , we say that  $C$  is a *positive vector*. The inclusion minimal vectors are called the *circuits* of  $\mathcal{M}(V)$ .

We described a vector  $C$  of  $\mathcal{M}(V)$  as a pair  $(C^+, C^-)$  of sets of indices of vectors in  $V$ . However, we will often abuse notation and identify  $C$ ,  $C^+$  and  $C^-$  with the vector subconfigurations  $V_C := \{v_i \in V \mid i \in C\}$ ,  $V_{C^+} := \{v_i \in V \mid i \in C^+\}$  and  $V_{C^-} := \{v_i \in V \mid i \in C^-\}$  respectively. Hence, we will use  $v_i \in C$  and  $i \in C$  interchangeably.

In this context, we will say that a subconfiguration  $W \subseteq V$  is a positive vector when there is a positive vector  $C$  with  $V_{C^+} = W$ . Observe that  $W$  is a positive vector if and only if the origin  $\mathbf{0}$  is contained in the relative interior of the convex hull of  $W$  (seen as points instead of vectors).

An (oriented) linear hyperplane  $h$  is defined by a normal vector  $v$  and corresponds to the set of points  $h := \{x \mid \langle v, x \rangle = 0\}$ . Its positive (resp. negative) side is the open halfspace  $h^+ := \{x \mid \langle v, x \rangle > 0\}$  (resp.  $h^- := \{x \mid \langle v, x \rangle < 0\}$ ). We denote by  $\bar{h}^+ = h \cup h^+$  and  $\bar{h}^- = h \cup h^-$  the corresponding closed halfspaces.

Fix a vector configuration  $V$ , and let  $h_1$  and  $h_2$  be linear hyperplanes with normal vectors  $v_1$  and  $v_2$ . We define their composition  $h_1 \circ h_2$  (with respect to  $V$ ) as a hyperplane  $h$  with normal vector  $v_1 + \varepsilon v_2$  for some very small  $\varepsilon$  whose value depends on  $V$ . Let  $v \in V$ . If  $\varepsilon$  is small enough, then  $v \in h$  if and only if  $v \in h_1 \cap h_2$ , and  $v \in h^\pm$  if and only if either  $v \in h_1^\pm$  or  $v \in h_1$  and  $v \in h_2^\pm$ .

## Gale duality

Let  $A := \{a_1, \dots, a_n\} \subset \mathbb{R}^d$  be a (multi-)point configuration whose affine span is  $\mathbb{R}^d$ . We denote by  $U$  the  $d \times n$  matrix whose column vectors contain the coordinates of the points of  $A$ . Observe that  $\text{rank}(U) = d$ . Let  $U'$  be the matrix  $\begin{pmatrix} U \\ \mathbf{1}^T \end{pmatrix}$  of homogenized coordinates of the points of  $A$ . Choose a basis  $(b_1, \dots, b_{n-d-1})$  of the kernel of  $U'$ , and denote by  $U^*$  the matrix whose column vectors are these  $b_i$ 's. In other words,  $U' U^* = \mathbf{0}$  and  $\text{rank}(U^*) = n - d - 1$ . Finally, define  $V = \{v_1, \dots, v_n\} \subset \mathbb{R}^{n-d-1}$  to be the row vectors of  $U^*$ . The sequence  $V$  is called a *Gale dual* of  $A$  and denoted  $A^*$ .

*Remark 2.1.* The Gale dual of  $A$  is well defined up to linear isomorphism.

The key property of Gale duality is that it translates affine evaluations into linear dependencies.

**Theorem 2.2.**  $\text{a-Val}(A) = \text{Dep}(A^*)$ .

In this equality, we are understanding  $A$  and  $A^*$  as sequences: the index of the point in the  $i$ th column of  $A$  must coincide with the index of the vector in the  $i$ th row of  $U^*$ . The following lemma is a particular case of this theorem.

**Lemma 2.3.** Let  $A := \{a_1, \dots, a_n\} \subset \mathbb{R}^d$  as before and  $A^* := \{v_1, \dots, v_n\} \subset \mathbb{R}^{n-d-1}$  denote its Gale dual. For any  $I \subset [n]$ , let  $F := \{a_i \mid i \in I\}$  and  $\bar{F}^* := \{v_i \mid i \notin I\}$ . Then:

(i)  $F$  is contained in a supporting hyperplane of  $A$  if and only if  $\bar{F}^\star$  contains a positive vector of  $\mathcal{M}(V)$ .

(ii)  $F$  are the only points contained in a supporting hyperplane of  $A$  if and only if  $\bar{F}^\star$  is a positive vector of  $\mathcal{M}(V)$ .

*Observation 2.4.* Observe that multiplying the vectors of  $V$  by positive scalars does not change its oriented matroid. Hence, we will often work with normalized versions of the Gale dual, which preserves the combinatorial type of the primal configuration.

*Example 2.5.* Consider the point configuration  $A$  in  $\mathbb{R}^2$  whose points' coordinates are encoded in the following matrix  $U$ , and its homogenized version  $U'$ :

$$U = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix}, \quad U' = \begin{bmatrix} a'_1 & a'_2 & a'_3 & a'_4 & a'_5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Its Gale dual is the vector configuration  $V = A^\star$  in  $\mathbb{R}^{n-d-1=2}$  defined by the matrix  $U^\star$ :

$$(U^\star)^\top = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ 1 & 0 & 0 & -2 & 1 \\ 1 & -2 & 1 & 0 & 0 \end{bmatrix}.$$

$A$  and  $V$  are shown in Figure 3, where one can check the assertions of Lemma 2.3. For example, the fact that  $\{a_1, a_4\}$  are contained in a common facet of  $\text{conv}(A)$  is reflected in the fact that  $\mathbf{0} \in \text{conv}(v_2, v_3, v_5)$ , but since  $\{a_1, a_4\}$  do not form a face,  $\mathbf{0} \notin \text{relint}(\text{conv}(v_2, v_3, v_5))$ .

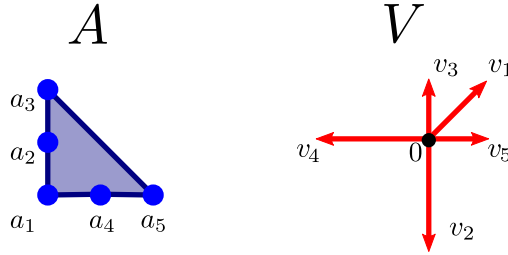


Figure 3: A point configuration  $A$ , and its Gale dual  $V$ .

### Deletion and Contraction

Two handy operations on vector configurations  $V$  are deletion and contraction.

The *deletion*  $V \setminus v$  of  $v \in V$  is the configuration  $V \setminus \{v\}$ . The *contraction*  $V/v$  of a non-zero vector  $v \in V$  is given by projecting  $V$  parallel to  $v$  onto some linear hyperplane that does not contain  $v$  and then deleting  $v$ . For example, one can use the map  $v_i \mapsto \tilde{v}_i := v_i - \frac{\langle v, v_i \rangle}{\langle v, v \rangle} v$ , and then  $V/v = \{\tilde{v}_i \mid v_i \neq v\}$  (see Figure 4 for an example). The contraction of  $\mathbf{0}$  is just its deletion. In terms of vectors of  $\mathcal{M}(V)$ ,

$$\begin{aligned}\mathcal{V}(V \setminus v) &= \{(C^+, C^-) \mid (C^+, C^-) \in \mathcal{V}(V), v \notin C^+ \cup C^-\}, \\ \mathcal{V}(V/v) &= \{(\tilde{C}^+, \tilde{C}^-) \mid \tilde{C}^+ = C^+ \setminus \{v\}, \tilde{C}^- = C^- \setminus \{v\}, \text{ for some } (C^+, C^-) \in \mathcal{V}(V)\},\end{aligned}$$

where the equalities of vectors  $C$  of  $V$  and vectors  $\tilde{C}$  of  $V/v$  in the previous statement should be understood in the sense that their elements have the same corresponding indices.

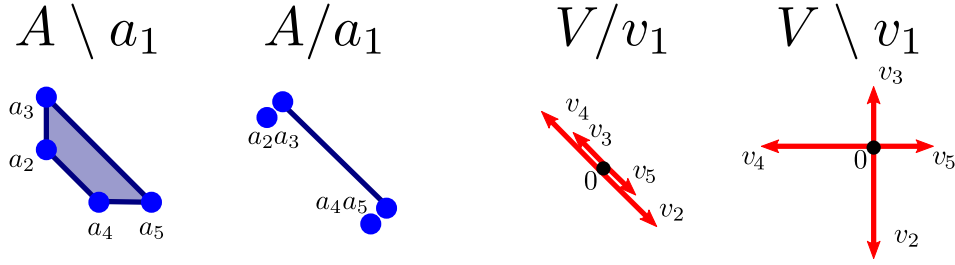


Figure 4: Example of contraction and deletion in the point and vector configurations  $A$  and  $V$  of Figure 3.

One can extend them to subsets  $W$  of  $V$ , just by iteratively deleting/contracting the elements of  $W$ . Moreover, these operations can also be defined in the primal for point configurations  $A$  just by homogenizing them (see Figure 4). In this context they are dual operations, since  $(A \setminus v)^* = A^*/v$  and  $(A/v)^* = A^* \setminus v$ .

## 2.2 The dual degree

We give a Gale dual interpretation of the degree, inspired by an analogue characterization for neighborly polytopes [28].

**Definition 2.6.** Let  $V$  be a vector configuration in  $\mathbb{R}^r$ . Its *dual degree* is

$$\deg^*(V) := \max_h |h^+ \cap V| - r,$$

where  $h$  runs through all linear hyperplanes of  $\mathbb{R}^r$ .

That is,  $\deg^*(V) = k$  if and only if  $k$  is the minimal integer such that for every linear hyperplane  $h$  there are at most  $r + k$  vectors of  $V$  in  $h^+$ .

Our first observation is that this definition is coherent with its primal counterpart.

**Proposition 2.7.**  $\deg(A) = \deg^*(A^*)$ .

*Proof.* Let  $A$  be a  $d$ -dimensional configuration of  $n$  points. By definition,  $\deg(A) = k$  if every subset  $S$  of  $A$  of size  $d - k$  is contained in a supporting hyperplane. Equivalently,  $\tilde{S}^*$  contains the origin in its convex hull for every  $\tilde{S}^*$  of size  $n - d + k = r + k + 1$  (see Lemma 2.3). Therefore, if  $\deg(A) = k$  there cannot be a hyperplane in  $\mathbb{R}^r$  through the origin that contains more than  $r + k$  vectors in  $h^+$ . Conversely, if there was a set of  $r + k + 1$  vectors whose convex hull does not contain the origin, then we could separate it from the origin by a hyperplane  $h$ .  $\square$

We also define the *dual codegree* of  $V$  as

$$\text{codeg}^*(V) := \min_h |\bar{h}^+ \cap V|,$$

with  $h$  running through all linear hyperplanes. Then, for  $|V| = r + d + 1$  it is easy to see that  $\text{codeg}^*(V) = d + 1 - \deg^*(V)$ , which is consistent with the primal definition.

*Observation 2.8.* With these definitions we can see the first connection with Tverberg theory mentioned in the introduction. Fix a vector configuration  $V$ , and let  $\bar{V}$  be the set of endpoints of the vectors in  $V$ . Observe that  $\text{codeg}^*(V) \geq k$  if and only if the origin has depth  $k$  in  $\bar{V}$ ,  $\mathbf{0} \in \mathcal{C}_k(\bar{V})$ .

The dual degree of a vector configuration (resp. degree of a point configuration) can only decrease when taking subconfigurations and contractions.

**Proposition 2.9.** For  $v \in V$ ,  $\deg^*(V \setminus v) \leq \deg^*(V)$  and  $\deg^*(V/v) \leq \deg^*(V)$ .

*Proof.* The first statement follows from  $|h \cap (V \setminus v)| \leq |h \cap V|$  for any hyperplane  $h$ .

For the second statement we can assume that  $v \neq \mathbf{0}$ . We have to show that no hyperplane  $h$  through  $v$  can have more than  $r + \delta - 1$  elements in its positive side, where  $r$  is the rank of  $V$  and  $\delta = \deg^*(V)$ . Let  $h_v$  be the hyperplane whose normal vector is  $v$ . If  $h^+ \cap V$  had more than  $r + \delta - 1$  elements in its positive side, then  $h \circ h_v$  would have at least the same elements and  $v$ , which would lead to a contradiction.  $\square$

**Corollary 2.10.** For  $a \in A$ ,  $\deg(A \setminus a) \leq \deg(A)$  and  $\deg(A/a) \leq \deg(A)$ .

### Pure vector configurations

Corollary 2.10 explains why we have to allow configurations that admit repeated points: even if  $A$  has no repeated points,  $A/a$  might (see the example of Figure 4).

However, it is straightforward to see that deleting repeated points of  $A$  does not change degrees neither weak Cayley configurations. This is the reason why we will usually only consider point configurations without repeated points.

**Lemma 2.11.** If  $a \in A$  is a repeated point, then  $\deg(A) = \deg(A \setminus a)$  and  $A$  is a (weak) Cayley configuration of length  $k$  if and only if  $A \setminus a$  is.

We define that a vector configuration  $V$  is *pure* if  $V^*$  does not have repeated points. To restrict to pure vector configurations, we will often use the following characterization.

**Lemma 2.12.** *A vector configuration  $V$  in  $\mathbb{R}^r$  is pure if and only if either  $r = 0$  or, for every linear hyperplane  $h$ ,  $|h^+ \cap V| \geq 2$  or  $|h^- \cap V| \geq 2$ .*

A first interesting consequence of this characterization is the following lemma which will allow us to settle the classification of point configurations of degree 0.

**Lemma 2.13.** *If  $V$  is a pure vector configuration in  $\mathbb{R}^r$  with  $r \neq 0$  then  $\deg^*(V) \geq 1$ .*

*Proof.* Let  $h$  be a hyperplane spanned by some subconfiguration  $W \subset V$ . By Lemma 2.12, we can assume that  $|h^+ \cap V| \geq 2$ . Then the contraction of  $V/W$  is a pure configuration of rank 1 that satisfies  $\deg^*(V/W) \geq 1$ . Now, the result follows from Proposition 2.9.  $\square$

**Proposition 2.14.** *A is a point configuration of degree 0 if and only if  $A$  is the set of vertices of a simplex (possibly with repetitions).*

*Proof.* Because of Corollary 2.10 and Lemma 2.11, it is enough to see that there are no  $d$ -dimensional point configurations of degree 0 with  $d+2$  points without repeated points, which follows from Lemma 2.13.  $\square$

**Corollary 2.15.**  *$V$  is a vector configuration of dual degree 0 if and only if it is a free sum of positive circuits, i.e.,  $\mathbb{R}^r$  is a direct sum of some subspaces  $U_1, \dots, U_l$  such that  $V$  is the union of positive circuits  $C_1, \dots, C_l$  with  $C_1 \subset U_1, \dots, C_l \subset U_l$ .*

### Irreducible vector configurations

In this dual setting some results mentioned in the introduction have a very easy interpretation. For example, Gale duals of pyramids are very easy to deal with. If  $A'$  is a pyramid over  $A$ , then  $A'^* = A^* \cup \{\mathbf{0}\}$ , adding the origin to  $A^*$ .

**Lemma 2.16.** *If  $A'$  is a pyramid over  $A$ , then  $\deg(A') = \deg(A)$ .*

*Proof.* Let  $V$  and  $V'$  be the Gale duals of  $A$  and  $A'$  respectively. Then, for every linear hyperplane  $h$ ,  $h^+ \cap V' = h^+ \cap V$ , and hence  $\deg^*(V) = \deg^*(V')$ .  $\square$

This motivates the following definition.

**Definition 2.17.** We say that a vector configuration  $V$  is *irreducible* if it does not contain the origin, that is, if  $V^*$  is not a pyramid.

Here is a simple observation on irreducible vector configurations.

**Proposition 2.18.** *An irreducible vector configuration  $V \in \mathbb{R}^r$  of dual degree  $\delta$  cannot contain more than  $2r + 2\delta$  vectors.*

*Proof.* Take any generic linear hyperplane  $h$ . By definition of the dual degree, there are at most  $r + \delta$  vectors in  $h^+$  or  $h^-$ .  $\square$

Its translation into the primal setting yields the proof of Theorem 1.4(III):

**Corollary 2.19.** *Let  $A$  be a point configuration in  $\mathbb{R}^d$  with  $n = r + d + 1$  points and degree  $\delta$ . If*

$$d \geq r + 2\delta$$

*then  $A$  must be a pyramid.*



### 2.3 Cayley<sup>★</sup> vector configurations

The concepts of Section 1.4.3 can be formulated in the Gale dual setting.

**Definitions 2.20.** Let  $V$  be a vector configuration in  $\mathbb{R}^r$ , then  $V$  is a

- *affine Cayley<sup>★</sup> configuration* of length  $k$ , if there exists a partition  $V = V_1 \uplus \dots \uplus V_k$  such that  $\sum_{v \in V_i} v = \mathbf{0}$  for  $i = 1 \dots k$ .
- *combinatorial Cayley<sup>★</sup> configuration* of length  $k$ , if there exists a partition  $V = V_1 \uplus \dots \uplus V_k$  such that for  $i = 1 \dots k$ ,  $V_i$  is a positive vector of  $\mathcal{M}(V)$ . That is, for each factor  $V_i$  there is a positive vector  $\lambda^{(i)} \in \mathbb{R}^{|V_i|}$  such that  $\sum_{v \in V_i} (\lambda^{(i)})_v \cdot v = \mathbf{0}$ .

These concepts coincide with their primal counterparts.

**Proposition 2.21.** *A is an affine (resp. combinatorial) Cayley configuration if and only if its Gale dual V is an affine (resp. combinatorial) Cayley<sup>★</sup> configuration.*

*Proof.* Let  $A$  be a combinatorial Cayley configuration of length  $k$ . That is, there exists a partition  $A = A_1 \uplus \dots \uplus A_r$ , such that for any  $\emptyset \neq I \subsetneq \{1, \dots, k\}$ ,  $\text{conv}(\bigcup_{i \in I} A_i)$  is a proper face of  $A$ . Let  $V = V_1 \uplus \dots \uplus V_k$  be the Gale dual of  $A$ ,  $V_i$  corresponding to  $A_i$ . Then, by Lemma 2.3,  $\overline{\bigcup_{i \in I} V_i} = \bigcup_{i \notin I} V_i$  is a positive vector of  $\mathcal{M}(V)$  for each  $\emptyset \neq I \subsetneq \{1, \dots, k\}$ . In particular, every  $V_i$  is a positive vector of  $\mathcal{M}(V)$ . Thus,  $V$  is a combinatorial Cayley<sup>★</sup> configuration. The converse is direct.

$A$  is an affine Cayley configuration of length  $k$  when there is an affine projection  $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^{k-1}$  that maps  $A$  onto the vertex set of a  $(k-1)$ -simplex  $\Delta$  with vertices  $\{w_1, \dots, w_k\}$ . Let  $A_i := A \cap \pi^{-1}(w_i)$  and observe that there is an affine function  $f_i$  such that  $f_i(a) = 1$  if  $a \in A_i$  and 0 otherwise. Let again  $V = \{v_1, \dots, v_n\}$  and  $A = \{a_1, \dots, a_n\}$  correspondingly. For  $i \in \{1, \dots, k\}$  we define  $V_i := \{v_j : a_j \in A_i\}$ . By duality, affine valuations on  $A$  correspond to linear dependences of  $V$ . Hence, we get that  $\sum_{j=1}^n f_i(a_j) v_j = \sum_{v \in V_i} v = \mathbf{0}$ . Thus,  $V$  is an affine Cayley<sup>★</sup> configuration with factors  $V_i$ . Again, the converse follows similarly.  $\square$

In this setting it is very easy to show that all combinatorial Cayley configurations can be realized by affine Cayley configurations. This motivates the use of the generic term Cayley configuration for combinatorial Cayley configurations.

*Proof of Proposition 1.12.* Let  $V = V_1 \uplus \dots \uplus V_k$  be the combinatorial Cayley<sup>★</sup> configuration whose Gale dual is  $A$ . Then, for each factor  $V_i$  there is a vector  $\lambda^{(i)}$  in  $\mathbb{R}^{|V_i|}$  such that  $\sum_{v \in V_i} (\lambda^{(i)})_v \cdot v = \mathbf{0}$ . Let  $\lambda$  be the vector in  $\mathbb{R}^{|V|}$  with entries  $\lambda_v = (\lambda^{(i)})_v$  if  $v \in V_i$ . Scaling the vectors of a vector configuration does not affect its combinatorics. Hence,  $\{\lambda_v v\}_{v \in V}$  is an affine Cayley<sup>★</sup> configuration combinatorially equivalent to  $V$ .  $\square$

#### Weak Cayley<sup>★</sup> configurations

Weak Cayley configurations also have a dual version.

**Definition 2.22.** A vector configuration  $V$  is a *weak Cayley<sup>\*</sup> configuration* of length  $k$ , if it contains  $k$  disjoint positive circuits of  $\mathcal{M}(V)$ .

While  $A$  is a weak Cayley configuration if and only if it contains a (possibly empty) subset  $A_0$  such that the contraction  $A/A_0$  is a combinatorial Cayley configuration of length  $k$ ,  $V$  is a weak Cayley<sup>\*</sup> configuration if and only if it contains a subset  $V_0$  such that the deletion  $V \setminus V_0$  is a combinatorial Cayley<sup>\*</sup> configuration of length  $k$ . Using this observation and Proposition 2.21, it follows directly from the duality of deletion and contraction that this definition is consistent with the primal version:

**Proposition 2.23.** *A is a weak Cayley configuration of length  $k$  if and only if its Gale dual  $A^*$  is a weak Cayley<sup>\*</sup> vector configuration of length  $k$ .*

We can now prove the estimation on the combinatorial degree of weak Cayley configurations.

*Proof of Proposition 1.3.* If  $V$  is a weak Cayley<sup>\*</sup> vector configuration in  $\mathbb{R}^r$  with factors  $V_1, \dots, V_k$ , then every linear hyperplane  $h$  contains at least one element of every factor in  $h^-$ . Therefore for any  $h$ ,  $|h^+ \cap V| \leq n - k$ , which proves that  $\deg^*(V) \leq n - r - k$ .  $\square$

We can see why Conjecture 1.5 is sharp.

*Example 2.24.* Let  $P$  be a neighborly polytope in even dimension  $d = 2m$  with  $n \geq 2d+1$  vertices. Using Theorem 1.6 on the facets of  $P$ , we see that  $P$  is simplicial. Therefore, its Gale dual  $V$  is a vector configuration in general position in  $\mathbb{R}^r$ , where  $2r+1 \geq n$ . Since  $P$  is neighborly,  $\delta := \deg^*(V) = \deg(\text{vert}(P)) = m$ . Regarding Conjecture 1.5 we see that  $V$  is a weak Cayley<sup>\*</sup> configuration of length  $d+1-2\delta = 1$ , but it cannot be a weak Cayley<sup>\*</sup> configuration of length 2. Indeed, since the vectors in  $V$  are in general position, each circuit  $C$  of  $V$  has cardinality  $r+1$ . And since  $n < 2r+2$ ,  $V$  cannot contain two disjoint circuits.

*Observation 2.25.* Observe that if  $V$  is a vector configuration and  $\bar{V}$  is its set of endpoints, then  $V$  is a weak Cayley<sup>\*</sup> configuration of length  $k$  if and only if  $\mathbf{0} \in \mathcal{D}_k(\bar{V})$ , i.e., the origin is a  $k$ -divisible point of  $\bar{V}$ . Together with Observation 2.8, this explains why Theorem 1.4(IV) is equivalent to Corollary 1.10.

## 2.4 Point configurations of degree 1

In this section we will prove the following result.

**Theorem 2.26.** *Let  $A$  be a  $d$ -dimensional point configuration. Then  $\deg(A) \leq 1$  if and only if one of the following two conditions hold:*

- (a) *A is a  $k$ -fold pyramid over a 2-dimensional point configuration without interior points or*
- (b) *A is a weak Cayley configuration of length  $d$ .*

Let us note that the dimension of each of the factors of a weak Cayley configuration of length  $d$  cannot be greater than 1, since they are included in a flag of faces of length  $d - 1$ . This shows that for  $d \geq 3$  the only weak Cayley configurations of length  $d$  are either  $k$ -fold pyramids over prisms over simplices with extra points on the “vertical” edges (in this case,  $A_0 = \emptyset$ , so  $A$  is a Cayley configuration of length 3) or simplices with a vertex  $v$  and points on the edges adjacent to  $v$  (here,  $A_0 = \{v\}$  and  $A/A_0$  is a Cayley configuration of length 3). This finishes the proof of Theorem 1.4(II). In particular, Theorem 2.26 implies the following corollary, which again motivates Conjecture 1.5.

**Corollary 2.27.** *Let  $A$  be a  $d$ -dimensional point configuration, if  $\deg(A) \leq 1$  then  $A$  is a weak Cayley configuration of length at least  $d - 1$ .*

The following proposition allows to relate the degree of the restriction of a vector configuration into a subspace and its contraction.

**Proposition 2.28.** *Let  $V$  be a vector configuration and let  $W \subset V$  be a subset of  $V$  such that  $\text{lin}(W) \cap V = W$ . If*

- $\text{rank } V = r$ ,  $|V| = r + d + 1$  and  $\deg^*(V) = \delta$ ,
- $\text{rank } W = r_W$ ,  $|W| = r_W + d_W + 1$  and  $\delta_W = \deg^*(W)$  (in  $\mathbb{R}^{r_W}$ ), and
- $\text{rank } V/W = r_{/W}$ ,  $|V/W| = r_{/W} + d_{/W} + 1$  and  $\delta_{/W} = \deg^*(V/W)$ ,

then

$$\begin{aligned} r &= r_W + r_{/W}, \\ d &= d_W + d_{/W} + 1, \\ \delta &\geq \delta_W + \delta_{/W}. \end{aligned}$$

*Proof.* By counting the number of elements in  $V$ , we get  $r + d + 1 = r_W + d_W + 1 + r_{/W} + d_{/W} + 1$ . And the identity  $r = r_W + r_{/W}$  implies that  $d = d_W + d_{/W} + 1$ .

Since the degree of  $W$  is  $\delta_W$ , there is an oriented hyperplane  $h_W$  of  $\text{lin}(W)$  that contains  $r_W + \delta_W$  elements of  $W$  in  $h_W^+$ . Let  $h'_W$  be a hyperplane of  $\mathbb{R}^r$  such that  $h'_W \cap \text{lin}W = h_W$  (such a hyperplane always exists, for example the only hyperplane that contains  $h_W$  and the orthogonal complement of  $\text{lin}(W)$ ). Since  $V/W$  has degree  $\delta_{/W}$ , there is an oriented hyperplane  $h_{/W}$  that contains  $\text{lin}(W)$  and that has  $r_{/W} + \delta_{/W}$  elements of  $V$  at  $h_{/W}^+$ . Then  $r + \delta \geq |(h_{/W} \circ h'_W) \cap V| = r_W + \delta_W + r_{/W} + \delta_{/W}$ . And therefore  $\delta_W + \delta_{/W} \leq \delta$ .  $\square$

Observe that we took the worst hyperplane in  $\mathbb{R}^r$  containing  $\text{lin}(W)$  and slightly perturbed it so that it cut  $\text{lin}(W)$  in its worst hyperplane.

In order to prove Theorem 2.26 we need the following crucial result about circuits in vector configurations of degree 1.

**Lemma 2.29.** *Let  $V$  be a pure vector configuration in  $\mathbb{R}^r$  with  $\deg^*(V) = 1$ . If  $C$  is a circuit of  $\mathcal{M}(V)$  with  $|C^+| > 0$  and  $|C^-| > 0$  then  $|C| = r + 1$ .*

*Proof.* Consider  $W = V \cap \text{lin}(C)$ . If  $|C^+| > 0$  and  $|C^-| > 0$ , then there is a hyperplane  $h$  in  $\text{lin}(C)$  with  $C \subset h^+$ . Therefore  $\deg^*(W) \geq 1$ . Since  $V$  is pure then  $V/W$  is also pure. If moreover  $|C| \leq r$ , then  $\text{rank}(V/W) \geq 1$ , and by Lemma 2.13,  $\deg^*(V/W) \geq 1$ . Now, Proposition 2.28 implies that  $\deg^*(V) \geq \deg^*(W) + \deg^*(V/W) = 2$ , which is a contradiction with  $\deg^*(V) = 1$ .  $\square$

This result yields some useful corollaries.

**Corollary 2.30.** *A pure vector configuration  $V$  in  $\mathbb{R}^r$  with  $r > 1$  and  $\deg^*(V) = 1$  cannot have repeated vectors.*

**Corollary 2.31.** *Let  $V$  be a pure vector configuration in  $\mathbb{R}^r$  with  $\deg^*(V) = 1$ . If  $C$  and  $D$  are different circuits of  $\mathcal{M}(V)$  with  $|C \cup D| \leq r + 1$ , then  $C \cap D = \emptyset$ .*

*Proof.* Observe first that due to Lemma 2.29, both  $C$  and  $D$  may be assumed to be positive circuits. Since  $C$  and  $D$  are minimal, there is some  $c \in C$  such that  $c \notin D$  and some  $d \in D$  such that  $d \notin C$ . If they are not disjoint, then there is some  $p \in (C \cap D)$ . Eliminating  $p$  on  $C$  and  $-D$  (oriented matroid circuit elimination), we find a circuit  $E$  with  $c \in E^+$ ,  $d \in E^-$  and size  $|E| \leq |C \cup D| - 1 \leq r$ . This contradicts Lemma 2.29.  $\square$

**Lemma 2.32.** *Let  $V$  be a pure vector configuration in  $\mathbb{R}^r$  with  $r + d + 1$  elements,  $d > 1$  and  $\deg^*(V) = 1$ . If  $V$  is a weak Cayley<sup>\*</sup> configuration of length  $d$  with factors  $C_1, \dots, C_d$ , and  $D$  is a circuit of  $\mathcal{M}(V)$  with  $|D| \leq r$ , then  $D = C_i$  for some  $1 \leq i \leq d$ .*

*Proof.* Assume that  $D \neq C_i$  for all  $1 \leq i \leq d$ . If there is some  $C_i$  with  $|C_i \cap D| = |C_i| - 1$ , then  $|C_i \cup D| \leq r + 1$  and we get a contradiction with Corollary 2.31. Otherwise, if  $|C_i \cap D| \leq |C_i| - 2$  for all  $i$  and  $|C_j \cap D| \neq \emptyset$  for some  $j$ , then  $|C_j \cup D| \leq n - (d - 1)2 = r + d + 1 - 2d + 2 = r - d + 3 \leq r + 1$  and we get again a contradiction with Corollary 2.31. Hence,  $D$  does not intersect any  $C_1, \dots, C_d$ . By Lemma 2.29,  $D$  can be assumed to be a positive circuit. Therefore,  $V$  is a weak Cayley<sup>\*</sup> configuration of length  $d + 1$ , so Proposition 1.3 implies that  $V$  has dual degree 0, a contradiction.  $\square$

In particular, in the situation of the previous lemma any subset  $W \subset V$  with  $|W| \leq r$  that does not contain any  $C_i$  must be linearly independent.

We finally have all the tools needed to prove the following proposition, which implies directly Theorem 2.26.

**Proposition 2.33.** *Let  $V$  be an irreducible pure vector configuration in  $\mathbb{R}^r$  with  $n = r + d + 1$  elements and  $d > 2$ . If  $\deg^*(V) = 1$  then  $V$  is a weak Cayley<sup>\*</sup> configuration of length  $d$ .*

*Proof.* The proof is by induction on  $r$ . If  $V$  is centrally symmetric (up to rescaling), then it must consist of  $r + 1$  pairs of antipodal vectors. Hence  $d = r + 1$  and  $V$  is a Cayley<sup>\*</sup> configuration of length  $d$ .

Otherwise, we use that  $V$  does not have multiple vectors by Corollary 2.30 in order to find some  $v \in V$  such that  $V \cap \text{lin}(v) = \{v\}$ . Now  $V/v$  is an irreducible pure vector configuration of rank  $r - 1$  with  $(r - 1) + d + 1 = r + d$  elements. Combining Proposition 2.28 and Lemma 2.13 we see that  $\deg^*(V) = 1$ . By induction hypothesis,  $V/v$  is a weak Cayley<sup>\*</sup> configuration with factors  $\tilde{C}_1, \dots, \tilde{C}_d$ . For convenience, we define  $\tilde{C}_0 := (V/v) \setminus \bigcup_{i=1}^d \tilde{C}_i$ .

By counting the number of elements in  $|V/v|$  we see that  $\sum_{i=0}^d |\tilde{C}_i| = r + d$ . This implies that

$$\sum_{i=1}^d (|\tilde{C}_i| - 2) \leq r - d,$$

and in particular  $|\tilde{C}_i| \leq r - 1$  for all  $1 \leq i \leq d$  because  $d > 2$  and  $|\tilde{C}_j| \geq 2$  for all  $j$ .

Each  $\tilde{C}_i$  (for  $1 \leq i \leq d$ ) is a positive circuit of  $\mathcal{M}(V/v)$  that expands to a circuit  $C_i$  of  $\mathcal{M}(V)$ , see the subsection on Deletion and Contraction in Section 2.1. By identifying corresponding elements we will consider from now on subsets of  $V/v$  as subsets of  $V \setminus \{v\}$ . In other words,  $\tilde{C}_i = C_i \setminus \{v\}$ . Since  $|C_i| \leq |\tilde{C}_i| + 1 \leq r$ , Lemma 2.29 shows that either  $v \notin C_i$  or  $v \in C_i^+$ . Hence,  $C_i$  is again a positive circuit with either  $C_i^+ = \tilde{C}_i^+$  or  $C_i^+ = \tilde{C}_i^+ \cup \{v\}$ . We will show that if some  $C_i$  contains  $v$ , then no other  $C_j$  can. This will prove our claim because then  $C_1, \dots, C_i, \dots, C_d$  will be the factors that make  $V$  a weak Cayley<sup>\*</sup> configuration.

For this, we will assume that  $v \in C_1$  and  $v \in C_2$  and reach a contradiction. We start with some definitions. For  $1 \leq i \leq d$ , let  $W_i$  be a subset of  $|\tilde{C}_i| - 2$  elements of  $\tilde{C}_i$ . We set  $W := \tilde{C}_0 \cup \bigcup_{i=1}^d W_i$ . We also choose  $v_1 \in \tilde{C}_1 \setminus W_1$  and  $v_2 \in \tilde{C}_2 \setminus W_2$ , and define  $W' := W \cup \{v_1, v_2\}$ .

A first observation is that the elements in  $W'$  must be linearly independent. Indeed, since  $|W'| = 2 + |\tilde{C}_0| + \sum_{i=1}^d (|\tilde{C}_i| - 2) = r + 2 - d \leq r - 1$ , their projections in  $V/v$  are already linearly independent because of Lemma 2.32. Now, let  $h'$  be a hyperplane through  $W'$  and otherwise in general position with respect to  $V$ . Observe that  $v \notin \text{lin}(W')$  because otherwise the vectors in  $W'$  would form a circuit in  $V/v$ . Therefore,  $v \notin h'$  and we can orient  $h'$  so that  $v \in h'^-$ . This implies that  $|h'^+ \cap \tilde{C}_1| = |h'^+ \cap C_1| = 1$  and  $|h'^+ \cap \tilde{C}_2| = |h'^+ \cap C_2| = 1$  because  $v \in C_1$  and  $v \in C_2$ . Moreover, since the elements in  $W'$  are linearly independent, we can perturb  $h'$  to a hyperplane  $h$  through  $W$  such that  $v_1, v_2 \in h^+$ . This yields  $|h^+ \cap \tilde{C}_1| = |h^+ \cap \tilde{C}_2| = 2$ .

Furthermore, we claim that  $|h^+ \cap \tilde{C}_i| \geq 1$  for all  $i > 2$ . Assume otherwise. Since  $v \in h^-$ , this would imply that there exists some  $i > 2$  such that  $C_i = \tilde{C}_i$  is completely contained in  $h$ , and hence, by construction, in  $\text{lin}(W)$ . In particular, some  $v_i \in \tilde{C}_i \setminus W_i$  satisfies  $v_i \notin W$  but  $v_i \in \text{lin}(W)$ . Therefore, this element would be part of a circuit in  $\{v_i\} \cup W$  (different from  $C_i$  since  $|C_i \cap W| = |C_i| - 2$ ). However,  $|C_i \cup W| \leq |W| + 3 \leq r$ , which would contradict Corollary 2.31.

Finally, let  $h''$  be any hyperplane such that  $W \subset h''^+$ . Now,  $|(h \circ h'')^+ \cap \tilde{C}_0| = |\tilde{C}_0|$ ,  $|(h \circ h'')^+ \cap \tilde{C}_i| = |\tilde{C}_i|$  for  $i = 1, 2$  and  $|(h \circ h'')^+ \cap \tilde{C}_i| \geq |\tilde{C}_i| - 1$  for  $2 < i \leq d$ . Therefore

$$|(h \circ h'')^+ \cap V| \geq 3 + \sum_{i=0}^d (|\tilde{C}_i| - 1) = r + 2.$$

This is a contradiction to  $\deg^*(V) = 1$ .  $\square$

## 2.5 Point configurations with $\deg(A) = k$

We will use Proposition 2.28 to prove Theorem 1.4(IV). Recall that in the dual setting the goal is to find many disjoint positive circuits. We will iteratively find a subconfiguration  $W$  of  $V$  of lower rank that has smaller dual degree. Eventually we will find a configuration of degree 0, and Corollary 2.15 will certify that in this subconfiguration there are already many disjoint positive circuits.

**Theorem 2.34.** *Let  $V$  be a vector configuration with  $r + d + 1$  elements and degree  $\delta := \deg^*(V)$ . Then  $V$  is a weak Cayley<sup>\*</sup> configuration of length at least  $d - 3\delta + 1$ .*

*Proof.* By Lemma 2.11, we can assume that  $V$  is pure. The proof will be by induction on  $\delta$ . We know that the result holds for  $\delta = 0$  because of Corollary 2.15.

Let  $h$  be any hyperplane spanned by elements of  $V$ . Let  $W = V \cap h$ . Then  $V/W$  is an irreducible pure vector configuration of rank 1 with  $d_W + 2$  elements and degree  $\delta_W := \deg^*(V/W)$ . By Lemma 2.13, we see that

$$\delta_W \geq 1.$$

From Proposition 2.18 we can deduce that  $(d_W - 2\delta_W) \leq r_W - 1 = 0$ . Therefore the previous equation implies that

$$(d_W - 3\delta_W) = (d_W - 2\delta_W) - \delta_W \leq -\delta_W \leq -1 \tag{2}$$

On the other hand,  $W$  is a vector configuration of rank  $r - 1$  with  $r + d_W$  elements and degree  $\delta_W := \deg^*(W)$ . By Proposition 2.28,

$$\delta_W \leq \delta - \delta_W \leq \delta - 1.$$

Moreover, again by Proposition 2.28 and (2)

$$d_W - 3\delta_W \geq (d - 3\delta) - (d_W - 3\delta_W) - 1 \geq (d - 3\delta).$$

Since  $\delta_W \leq \delta - 1$  we can apply induction to see that  $W$  contains  $d_W - 3\delta_W + 1 \geq d - 3\delta + 1$  disjoint positive circuits, and hence so does  $V$ .  $\square$

Of course, this theorem is just a first step, since it only proves that there is a subspace that contains many disjoint circuits, but ignores the vectors outside of this subspace, which could form more disjoint circuits. Some preliminary results we obtained show that it should be possible to improve on this in future work. Note that there is not even yet a linear bound for the Ehrhart-theoretical counterpart of this theorem (see statement (iv) in Section 1.4.4).

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